Chapter 5. Partitions

1. Compositions.

Defn. A sequence $(a_1, a_2, \dots, a_K)$ of nonnegative integers s.t. $a_1 + a_2 + \cdots + a_K = n$ is called a **weak composition** of n.

If, in addition, the a_i are positive, then the sequence $(a_1, a_2, \dots, a_K)$ is called a **composition** of n.

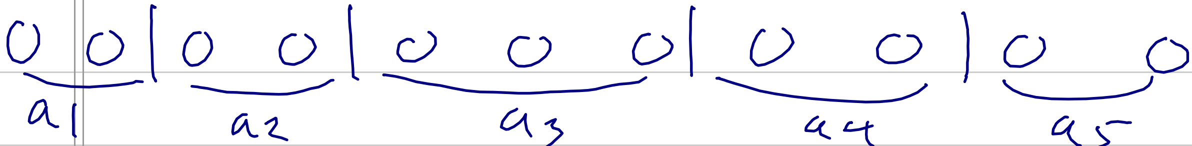
Example 5.1. The number of weak compositions of n into k parts is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Proof:

By adding 1 to each a_i, we have a bijection between weak compositions of n and compositions of $n+k$.

We will count the latter. The number of composition of $n+k$ is exactly the way to divide a row of $n+k$ identical balls into k non empty parts.



We can divide this row of ball by using $k-1$ sticks. These stick can be placed at $n+k-1$

gap between two consecutive balls. Finally, there are $\binom{n+k-1}{k-1}$ way to place $k-1$ sticks in $n+k-1$ gaps. $\square$

Corollary: The number of compositions of n into k parts is $\binom{n-1}{k-1}$.

Corollary: The number of all compositions of n is 2^{n-1}.

Solution: $\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$.

2. Set Partitions

Defn A "partition" of the set $[n]$ is a collection of non-empty subsets so that each element of $[n]$ belongs to exactly one of the subsets

The number of partitions of $[n]$ into k non-empty subsets is denoted by $S(n,k)$.
The number $S(n,k)$ is called the **Stirling number of the second kind**.

Example 5.2. $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$

Proof. We classify the set of partitions of

n into k subsets.

1) One subset is $\{n\}$
2) There no subset $\{n\}$

Type one has $S(n-1, k-1)$ partitions.

For each partition of type 2, n must be in one of k subsets of a partition of $[n-1]$ into k parts.
$\Rightarrow$ Type 2 has $k\, S(n-1, k)$ partitions.

$\square$

Example 5.3. Find the number of surjection $f: [n] \to [k]$.

Solution. Let f be a surjective $[n] \to [k]$
$\Rightarrow$

$(f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k))$ form a <u>ordered</u> partition of $[n]$ into k parts.

Since each non-ordered partition of n into k parts can generate $k!$ ordered partitions (by labelling the k subsets), the number of ordered partitions is $k!\, S(n, k)$ $\square$

Example 5.4. Prove that for all real numbers x, and all non-negative integers n

$$x^n = \sum_{k=0}^{n} S(n,k)\, (x)_k$$

Where $(x)_k = x(x-1)\dots(x-k+1)$.

Proof. Both sides are polynomials of x with degree n.

We will show that the two polynomials agree for all positive integer x.

LHS is the number of all function $[n] \to [x]$

We NTS that the RHS is the same:

We consider the set $f([n]) = I$.

If $|I| = k$, then there are $\binom{x}{k}$ choices for the image I, then by the previous example there are $k!\, S(n,k)$ choices for the surjective function $f_I : [n] \to I$.

As $(x)_k = k!\binom{x}{k}$, we have the number of all function $f : [n] \to [x]$ is $\sum_k (x)_k\, S(n,k)$. $\square$

Defn. The number of all partitions of $[n]$ into non-empty subsets is denoted by $B(n)$, and called the nth **Bell number.** We also set $B(0) = 1$.

Thus

$$B(n) = \sum_{i=0}^{n} S(n,i).$$

Example 5.5. Prove that

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

Proof. We need to show that RHS counts all partitions of $[n+1]$.

Assume that the element $n+1$ is in a subset of size $n-i+1$. Then there are i elements that are not in the same subset as $n+1$. Therefore, there are $\binom{n}{i}$ ways to choose such i elements, and there are $B(i)$ way to partition them.
Summing over all possible values of i, we are done. $\square$

3. Integer Partitions.

Defn. Let $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_K \geq 1$ be integers such that $a_1 + a_2 + \dots + a_K = n$.
Then the sequence $(a_1, a_2, \dots, a_K)$ is called a **partition** of the integer n.
The number of all partitions of n is denoted by $p(n)$.
The number of all partitions of n into K part is denoted by $p_K(n)$.

Example: $p(5) = 7$:

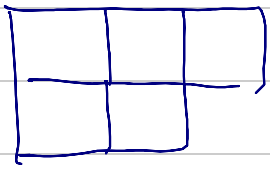
(5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, and $(1,1,1,1,1)$.

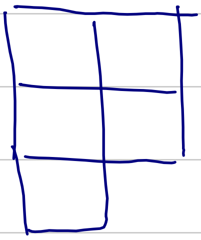
We have

$$p(n) \sim \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Defn. A "Ferrers diagram" (Shape) of a partition $p = (x_1, x_2, \ldots, x_k)$ is a set of n square boxes forming k rows, such that all rows start at the same vertical line, and row i has x_i boxes

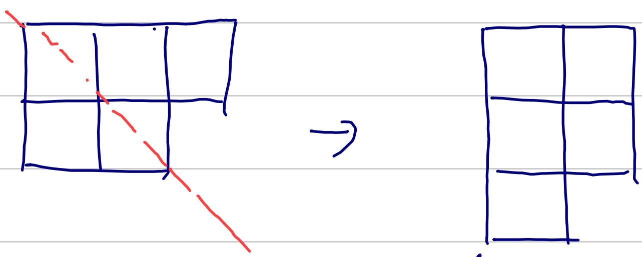
Example

$(3,2)$ = 

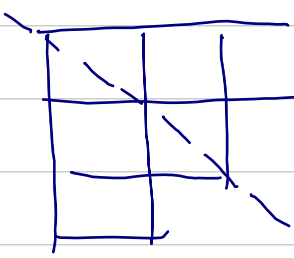
$(2,2,1)$ = 

Defn. If we reflect a Ferrers diagram of a partition p with respect to it main diagonal, we obtain a Ferrers diagram of another partition p'. We call p' the **conjugate partition** of p.

Example:

$p = (3,2)$  $= p' = (2,2,1)$

Defn. A partition of n is called **self-conjugate** if it equal to its conjugate

$(3,2,1)$ = 

Example 5.6. The number of partitions of n into atmost K parts is equal to that of partitions of n into parts not larger than K.

Proof.

Let μ be a partition of n into atmost K parts. The first column in a Ferrers diagram of μ is at most K.

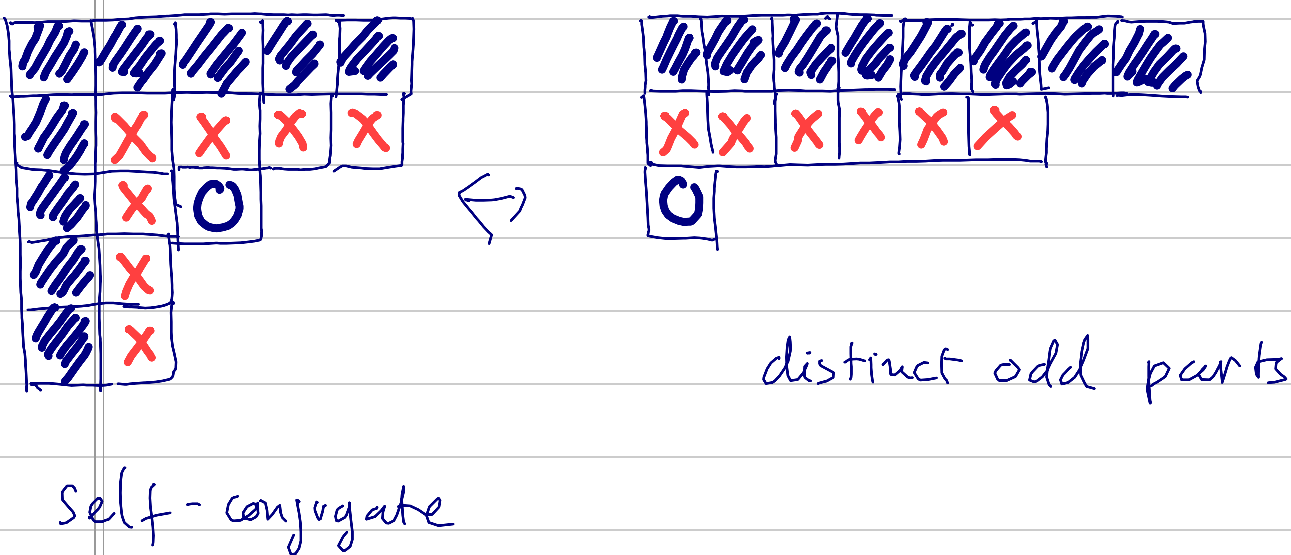
Consider the conjugate μ' of μ. The Ferrers diagram of μ' has the first row at most K.

This mean the conjugation gives a bijection between two set of partitions. $\square$

Example 5.7. The number of partitions of n into n distinct odd parts is equal to that of all self-conjugate partitions of n.

Proof:

Consider the following bijection



$\square$

Example 5.8. Let $q(n)$ be the number of partitions of n in which each part is at least 2. Prove that $q(n) = p(n) - p(n-1)$, for all $n \geqslant 2$.

Proof:

We need to show

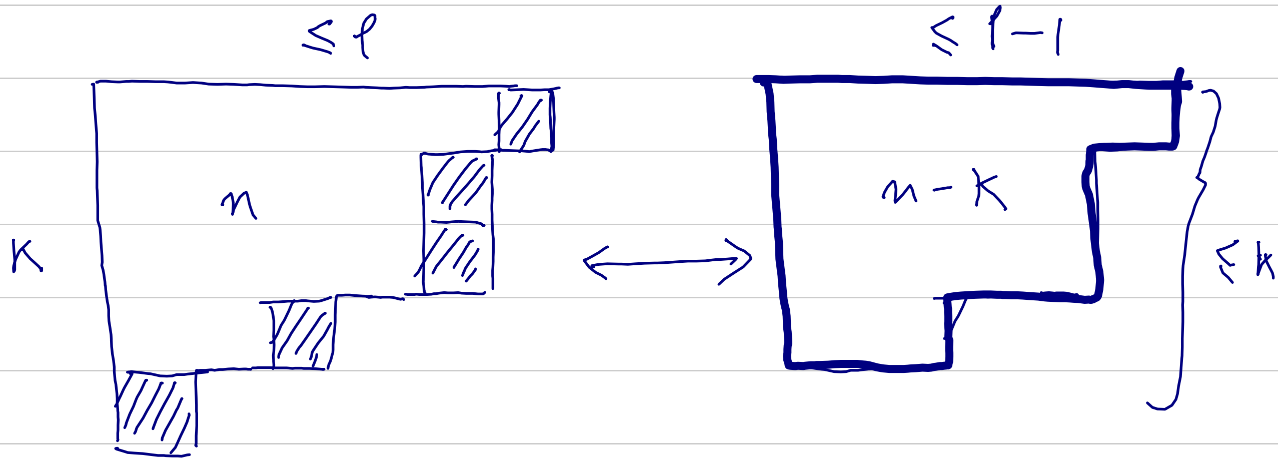
$$\underbrace{p(n) - q(n)}_{\substack{\#\text{ partitions of } n \\ \text{in which there} \\ \text{is a part } 1}} = \underbrace{p(n-1)}_{\substack{\#\text{ partitions} \\ \text{of } n-1}}$$

bijection: remove a part 1 $\square$

Example 5.9. Denote by $p_{k,\ell}(n)$ the number of partitions of n into at most k parts, and all parts are at most ℓ. Prove that

$$p_{k,\ell}(n) = p_{k-1,\ell}(n) + p_{k,\ell-1}(n-k).$$

Hint:



$\square$

Example 5.10. Prove that

$$P_{k,\ell}(n) = \binom{k+\ell}{\ell}.$$

Proof: We prove by induction on $k+l$.

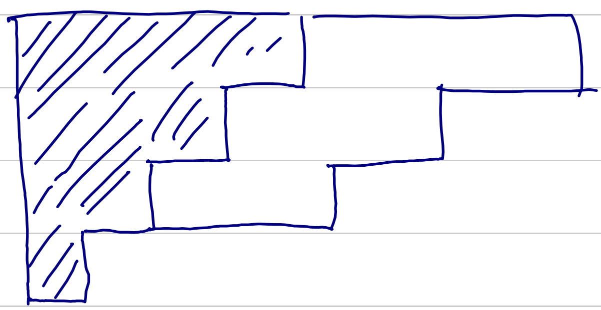
$$P_{k,\ell}(n) = P_{k-1,\ell}(n) + P_{k,\ell-1}(n-k)$$

By induction hypothesis the RHS is equal to

$$\binom{k+l-1}{\ell} + \binom{k+l-1}{\ell-1} = \binom{k+\ell}{\ell}. \quad \square$$

Example 5.11. Prove that the number of partitions of n into k distinct parts is equal to the number of partitions of $n - \frac{k(k+1)}{2}$ into at most k parts.

Hint:



$\square$

Connection between two types of partitions

Let $\Pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ be a set partition of $[n]$. If we arrange the sequence of subset sizes $|\pi_1|, |\pi_2|, \dots, |\pi_k|$ in nonincreasing order to get a seq. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$ is an integer partition of n. We say λ is the **type** of the set partition π.

Theorem. Let $\lambda = (\lambda_1, \ldots, \lambda_K)$ be an integer partition of n, in which part i appear m_i times. Then the number of set partitions of type λ is

$$\frac{\binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_K}}{m_1! \, m_2! \cdots m_j!}$$

Proof:

Take λ_i balls of color i ($i \in [K]$). Arrange them in a row by $\binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_K}$ different ways. Consider the following map f between these linear orderings and set partitions of $[n]$ with type λ.

For each ordering γ we define a set partition $\pi = f(\gamma)$ as follows: Two number i and j are in the same subset if the balls at positions i and j in the ordering γ have the same color.

It is easy to see that f is onto. However it is not 1-1. Each set partition of type λ corresponds to excactly $\prod_{i \geq 1} m_i!$.

Indeed, for example if $\lambda_1 = \lambda_2$, then having λ_1 balls of color 1 in a subset A of positions, and having λ_1 balls of color 2 in a subset B of positions will result in the same partition as having λ_1 balls of color 2 in A and λ_1 balls of color 2 in B.

In general, if m_i is the multiplicity of i in λ, then there are $m_i!$ ways the m_i color classes having i balls each can be permuted amg each other.

Thus, # of set partitions of type λ is

$$\frac{1}{\prod_i m_i!} \times \# \text{ of linear orderings.}$$

$$= \frac{\binom{n}{a_1, a_2, \ldots, a_K}}{m_1!\, m_2! \ldots m_K!}. \qquad \square$$

Theorem (Euler)

The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Proof (Glaisher)

Let $\lambda = (1^{m_1} 3^{m_3} 5^{m_5} \ldots)$ be a partition of n into odd parts. For every i, let $\varphi(\lambda)$ contain part $i \cdot 2^r$, iff m_i written in binary has 1 at the r-th position.

$\Rightarrow \varphi : \mathcal{O}_n \to \mathcal{D}_n$

Define $\psi : D_n \to O_n$ as follows. Start with $\mu = (\mu_1, \mu_2, \dots) \in D_n$. Subsitute every even part μ_i by two parts $\frac{\mu_i}{2}$. Repeat until the resulting partition λ has no even part. Set $\psi(\mu) = \lambda$.

Exercise: check that φ is well define, and $\varphi^{-1} = \psi$.

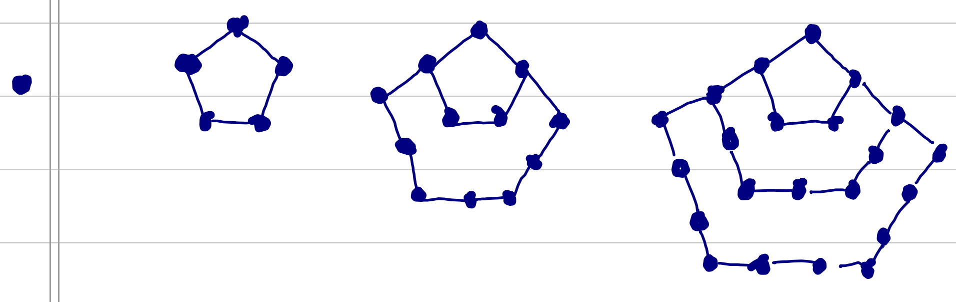
Example 5.12. Euler used the following recurrence to calculate $p(n)$:

+ Pentagonal number is an integer of the form $k(3k \pm 1)/2$. Example: 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, ..

+ $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \dots$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \left[p\left(n - \frac{m(3m-1)}{2}\right) + p\left(n - \frac{m(3m+1)}{2}\right) \right].$$

This is called "Euler's Pentagonal Theorem".



$\frac{3n^2 - n}{2}$

Example 5.13. (Rojers- Ramanujan)

The number of partitions of n into parts $\equiv \pm 1 \pmod 5$ is equal to the number of partitions of n into parts which differ by at least 2.

$$\iff \quad 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{i=0}^{\infty} \frac{1}{\left(1-q^{5(i+1)}\right)\left(1-q^{5i+4}\right)}$$

People are still seeking for a direct bijective proof of the identity.

Note that the theorem should be named as **Rojers - Ramanujan-Schur Theorem**. Only Rojers and Schur gave proof for this beautiful formula. Ramanujan came up with the indentity but couldn't give a proof. However, Hardy (a Ramanujan's closed friend) named the theorem after Rojers and Ramanujan.